SOLUTION EXERCISE SHEET 18

Exercise 1. We begin by applying the substitution $x = \rho^2$ which yields

$$\int_0^\infty \frac{\sqrt{x}}{x^2 + a^2} dx = 2 \int_0^\infty \frac{\rho^2}{\rho^4 + a^2} d\rho = \int_{-\infty}^\infty \frac{\rho^2}{\rho^4 + a^2} d\rho$$

where the last step follows from the evenness of the integrand. Consider now the function

$$f(z) = \frac{z^2}{z^4 + a^2}.$$

This function has simple poles at

$$z_j = \sqrt{a} e^{i(\frac{\pi}{4} + j\frac{\pi}{4})}, \qquad j = 0, 1, 2, 3.$$

Two of these lie in the upper half-plane, namely z_0 and z_1. The respective residues are given by

$$\operatorname{Res}(f, z_j) = \frac{z_j^2}{(z^4 + a^2)'|_{z=z_j}} = \frac{1}{4z_j}$$

Therefore, by using the standard semicircle argument (see the solutions to exercise sheet 17 for the details), we conclude that

$$\int_0^\infty \frac{\sqrt{x}}{x^2 + a^2} dx = \frac{\pi i}{2} \left(\frac{1}{z_0} + \frac{1}{z_1} \right).$$

Lastly, we compute that

$$\frac{1}{z_0} + \frac{1}{z_1} = \frac{\sqrt{2}}{\sqrt{a}} \left((1+i)^{-1} + (-1+i)^{-1} \right) = \frac{-i\sqrt{2}}{\sqrt{a}}$$

which finally yields the result

$$\int_0^\infty \frac{\sqrt{x}}{x^2 + a^2} dx = \frac{\pi}{\sqrt{2a}}.$$

For the second integral, we perform the same substitution as above to obtain

$$\int_0^\infty \frac{\sqrt{x}}{(x^2 + a^2)^2} dx = \int_{-\infty}^\infty \frac{\rho^2}{(\rho^4 + a^2)^2} d\rho.$$

The difference compared to the first integral is that the poles are no longer simple but instead of second order. Therefore, we employ the formula

$$\operatorname{Res}(f, z_j) = \lim_{z \to z_j} \frac{d}{dz} ((z - z_j)^2 f(z)) \tag{1}$$

with $f(z) = \frac{z^2}{(z^4+a^2)^2}$ and z_j as above. To do this in a concise manner, we first make some further considerations. To that end, we begin by remarking that performing a

Taylor expansion of $g(z) = (z^4 + a^2)$ near $z = z_j$ yields that

$$g(z) = g'(z_j)(z - z_j) + \frac{g''(z_j)}{2}(z - z_j)^2 + O((z - z_j)^3).$$

Plugging this into (1) yields

$$\begin{aligned}\lim_{z\to z_j} \frac{d}{dz}((z - z_j)^2 f(z)) &= \lim_{z\to z_j} \frac{d}{dz}\left(\frac{z^2}{g'(z_j)^2 + g'(z_j)g''(z_j)(z - z_j) + O((z - z_j)^2)}\right)\\ &= \frac{2z_j}{g'(z_j)^2} - \lim_{z\to z_j}\left(\frac{z^2 g'(z_j)g''(z_j)}{(g'(z_j)^2 + g'(z_j)g''(z_j)(z - z_j) + O((z - z_j)^2))^2}\right)\\ &= \frac{2z_j}{g'(z_j)^2} - \frac{z_j^2 g''(z_j)}{g'(z_j)^3}\\ &= \frac{1}{8z_j^5} - \frac{3}{16z_j^5} = -\frac{1}{16z_j^5}\end{aligned}$$

So, given that $z_j^5 = a^2 z_j$, we compute that

$$\int_0^\infty \frac{\sqrt{x}}{(x^2 + a^2)^2}dx = 2\pi i\left(-\frac{1}{16z_1^5} - \frac{1}{16z_0^5}\right) = -\frac{\pi i}{8a^{\frac{5}{2}}}\left(\frac{1}{z_0} + \frac{1}{z_1}\right) = \frac{\pi}{4\sqrt{2}a^{\frac{5}{2}}}.$$

Exercise 2. Recall that the Fourier transform of f is defined as

$$\widehat{f}(\xi) = \int_{\mathbb{R}} e^{-ix\xi} f(x)dx.$$

Now, for $z \in \mathbb{C}$, consider the function

$$g(z) = \int_{\mathbb{R}} e^{-ixz} f(x)dx.$$

We claim that this is a well-defined continuous function. To see that, let $I = [-N, N]$ be a closed interval such that the support of f is contained in I. Additionally, we let M be the maximum of f. Moreover, we note that for any $z = \xi + i\omega \in \mathbb{C}$ the estimate

$$\sup_{x\in I}\left|e^{-ixz}\right| = \sup_{x\in I}|e^{x\omega}| \leq 1 + e^{N\omega}$$

holds. So,

$$|g(z)| = \left|\int_{\mathbb{R}} e^{-ixz} f(x)dx\right| M \leq \int_{-N}^{N} |e^{-ixz}|dx \leq 2MN(1 + e^{N\omega}).$$

Furthermore, since integrating a function which depends continuously on parameters over a compact set is again a continuous function in said parameters, it follows that g is in fact a continuous function that extends $\widehat{f}$ to the whole complex plane.

There are several ways to check the holomorphicity of g. One such way is to use the Wirtinger derivative $\overline{\partial}$ (with respect to the variable z). From the Leibnitz integral rule it follows that

$$\overline{\partial} g(z) = \int_{-N}^{N} \overline{\partial} e^{-ixz} f(x)dx = 0$$

as

$$\overline{\partial} e^{-ixz} = 0.$$

To show the last claim, assume that we are given a $f \in C_c^0(\mathbb{R})$ with $\widehat{f} \in C_c^0(\mathbb{R})$ and let $\widehat{I} = [-\widehat{N}, \widehat{N}]$ be such that *supp*$(\widehat{f}) \subset I$. Then, by the above considerations, $\widehat{f}$ extends to an entire function that vanishes on the set $[\widehat{N}+1, \widehat{N}+2]$. Hence, by the identity theorem, $\widehat{f}$ needs to identically vanish.

Exercise 3. For the first statement, we note that the exponential decay allows us to extend $\widehat{f}$ to the desired strip, as for any $z \in S_a := \{z \in \mathbb{C} : |\operatorname{Im}(z)| < a\}$ we have that

$$\left| \int_{-\infty}^{\infty} e^{-ixz} f(x) dx \right| \leq \int_{-\infty}^{\infty} \left| e^{\operatorname{Im}(z)x} f(x) \right| dx \leq \int_{-\infty}^{\infty} e^{-(|a| - \operatorname{Im}(z))x} dx < \infty.$$

To show that $\widehat{f}$ is continuous on the strip S_a, we let $z_1, z_2 \in S_a$ and note that this implies that there exists a b with $0 \leq b < a$ such that

$$\begin{aligned} |e^{-ixz_1} - e^{-ixz_2}| &= \left| -ix \int_{z_2}^{z_1} e^{-ixy} dy \right| \\ &\leq x|z_1 - z_2| \sup_{\operatorname{Im}(z) \in [\operatorname{Im}(z_1), \operatorname{Im}(z_2)]} e^{|x| \operatorname{Im}(z)} \\ &\leq x|z_1 - z_2| e^{|x|b} \end{aligned}$$

Hence,

$$\begin{aligned} |\widehat{f}(z_1) - \widehat{f}(z_2)| &= \left| \int_{-\infty}^{\infty} (e^{-ixz_1} - e^{-ixz_2}) f(x) dx \right| \\ &\leq |z_1 - z_2| \int_{-\infty}^{\infty} x e^{-|x|(a-b)} \\ &\leq C_b |z_1 - z_2| \end{aligned}$$

where C_b is some constant that depends on b. Therefore, we see that $\widehat{f}$ is a continuous function on the strip S_a. To show that $\widehat{f}$ is holomorphic, we will use Morera's Theorem. Hence, let γ be any closed C^1 curve in S_a. If we can show that we are allowed to interchange the order of integration, then we are done since in that case

$$\int_\gamma \widehat{f}(z) dz = \int_\gamma \int_{-\infty}^{\infty} e^{-ixz} f(x) dx dz = \int_{-\infty}^{\infty} f(x) \int_\gamma e^{-ixz} dz dx = 0.$$

Since the image of γ is a compact set contained in S_a, we know that there exists a c with $0 \leq c \leq a$ and $C > 0$ such that

$$\sup_{\gamma(t)} \left| e^{-ix\gamma(t)} \gamma'(t) \right| \leq C e^{|x|c}.$$

From this estimate it follows that we can apply Fubini's theorem to conclude the first part of this exercise.

The idea of proving the second statement is to deform the contour of integration. To that end, we first note that, as by assumption,

$$\lim_{|\xi|\to\infty} \xi^2 \max_{b\in[-a,a]} \widehat{f}(\xi+ib) = 0,$$

and as $\widehat{f}$ is continuous on the closure of the strip S_a there exists a constant $C > 0$ such that

$$|\widehat{f}(z)| \leq \frac{C}{1+\xi^2}$$

for all $z \in \mathbb{C}$ with $\mathrm{Im}(z) \in [-a, a]$. Next, we make a case distinction on the sign of x. We start with the case $x \geq 0$. Then, for $\varepsilon > 0$ small, consider the curve

$$\gamma_R(t) := \begin{cases} \gamma_{R,1}(t) := -R + 2tR & \text{for } t \in [0,1] \\ \gamma_{R,2}(t) := R + (t-1)(a-\varepsilon)i & \text{for } t \in [1,2] \\ \gamma_{R,3}(t) := (a-\varepsilon)i + R - 2(t-2)R & \text{for } t \in [2,3] \\ \gamma_{R,4}(t) := -R + (a-\varepsilon)i - (t-3)(a-\varepsilon)i & \text{for } t \in [3,4]. \end{cases}$$

Since the function $\widehat{f}(z)e^{izx}$ is holomorphic on the stated strip, for every $x \in \mathbb{R}$, it follows that

$$\int_{\gamma_R} \widehat{f}(z)e^{izx} dz = 0.$$

Moreover, we compute that

$$\begin{aligned} \left| \int_{\gamma_{R,2}} \widehat{f}(z)e^{izx} dz \right| &= \left| \int_1^2 i(a-\varepsilon)\widehat{f}(R + (t-1)(a-\varepsilon)i)e^{ix(R+(t-1)(a-\varepsilon)i)} dt \right| \\ &\leq a\frac{C}{1+R^2}. \end{aligned}$$

Similarly, it follows that

$$\left| \int_{\gamma_{R,4}} \widehat{f}(z)e^{izx} dz \right| \leq a\frac{C}{1+R^2}$$

Thus, by taking the limit $\mathbb{R} \to \infty$, we observe that

$$f(x) = \int_{-\infty}^{\infty} \widehat{f}(\xi)e^{ix\xi} d\xi = \int_{-\infty}^{\infty} \widehat{f}(\xi + i(a-\varepsilon))e^{ix(\xi+i(a-\varepsilon))} d\xi.$$

Therefore,

$$|f(x)| \leq e^{-x(a-\varepsilon))} \int_{-\infty}^{\infty} \frac{C}{1+\xi^2} d\xi \leq C\pi e^{-x(a-\varepsilon)}.$$

Since ε was arbitrary, it follows that

$$|f(x)| \leq C\pi e^{-x(a-\varepsilon)}$$

for all $x \geq 0$ and $\varepsilon > 0$, which in turn implies that

$$|f(x)| \leq C\pi e^{-ax}$$

for all $x \geq 0$. For negative x one argues in the same fashion with the sole difference that instead of γ_R one uses $\widetilde{\gamma}_R$, which is defined as

$$\widetilde{\gamma}_R(t) := \begin{cases} \widetilde{\gamma}_{R,1}(t) := -R + 2tR & \text{for } t \in [0,1] \\ \widetilde{\gamma}_{R,2}(t) := R - (t-1)(a-\varepsilon)i & \text{for } t \in [1,2] \\ \widetilde{\gamma}_{R,3}(t) := -(a-\varepsilon)i + R - 2(t-2)R & \text{for } t \in [2,3] \\ \widetilde{\gamma}_{R,4}(t) := -R - (a-\varepsilon)i + (t-3)(a-\varepsilon)i & \text{for } t \in [3,4], \end{cases} \tag{2}$$

i. e. one integrates along a rectangle in the lower half-plane.

Exercise 4. Since log is increasing on the positive real axis, the inequality

$$\theta(x) = \sum_{p<x,\, p\, prime} \log(p) \leq \sum_{p<x,\, p\, prime} \log(x) = \pi(x)\log(x) \tag{3}$$

holds. For the other direction, we note that for any $\varepsilon > 0$ we have the estimate

$$\begin{aligned} \theta(x) &\geq \sum_{x^{1-\varepsilon} \leq p < x,\, p\, prime} \log(p) \geq (1-\varepsilon) \sum_{x^{1-\varepsilon} \leq p < x,\, p\, prime} \log(x) \\ &\geq (1-\varepsilon)[\pi(x) - \pi(x^{1-\varepsilon})]\log(x) \\ &= (1-\varepsilon)\pi(x)\log(x)\left(1 - \frac{\pi(x^{1-\varepsilon})}{\pi(x)}\right). \end{aligned}$$

Furthermore, as $\pi(x^{1-\varepsilon}) \leq x^{1-\varepsilon}$, which we combine with the previously established inequality

$$\theta(x) \leq \pi(x)\log(x)$$

to conclude that

$$0 \geq -\frac{\pi(x^{1-\varepsilon})}{\pi(x)} \geq -\frac{x^{1-\varepsilon}}{\pi(x)} \geq -\frac{x^{1-\varepsilon}\log(x)}{\theta(x)}.$$

Consequently, as

$$\lim_{x\to\infty} \frac{x}{\theta(x)} = 1,$$

we readily infer that

$$\lim_{x\to\infty} -\frac{\pi(x^{1-\varepsilon})}{\pi(x)} = 0.$$

Hence, we see that for every $\varepsilon \geq 0$ there exists a x_ε such that for all $x \geq x_\varepsilon$ we have that

$$\theta(x) \geq (1-\varepsilon)^2\pi(x)\log(x) \geq (1-\varepsilon)^2\theta(x).$$

Then, since ε was arbitrary, the claim follows.